Technical Comments

Comment on "Spin Effects on Rocket Nozzle Performance"

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Nomenclature

a = sonic velocity
$A_{st, perp}$ = area of stream tube measured perpendicular to nozzle axis
g = gravitational const
K = constant in isentropic relationship relating P and ρ
$\dot{m}$ = mass flow rate
P = pressure
r = radius
R = chamber radius or gas const
T = temperature
V = velocity
γ = specific heat ratio
ρ = density
ω = angular velocity

Subscripts

a, t = axial and tangential direction, respectively
0 = conditions in motor chamber before converging section
$*$ = conditions at nozzle throat

Introduction

THIS comment actually refers to two notes by Manda[1] and Bastress.[2] The models treated by them and by this author in an unpublished work are essentially the same. The treatments of this model and the results, however, differ in each case. The basic assumptions of the model (or results of equivalent assumptions) other than such usual assumptions as inviscid flow and ideal gas behavior are as follows: 1) The swirling flow maintains a "spinning plug" (or forced vortex) pattern as it passes through the nozzle. 2) Angular momentum is conserved.

Comparisons with Bastress and Manda

The results obtained by Bastress, Manda, and this author for the square of the axial velocity at the throat are:

Bastress

$$V_{a_*}^2 = 2g\ \gamma RT_0/(\gamma + 1) - \omega_0^2[(R_0/R_*)^2 - 1]^2 r^2 \quad (1)$$

Manda

$$V_{a_*}^2 = 2g\ \gamma RT_0/(\gamma + 1) - \omega_0^2\ (R_0/R_*)^2\ [2(R_0/R_*)^2 - 1]r^2 \quad (2)$$

[It should be noted that Bastress' and Manda's a_*^2 refer to sonic velocity at the throat under nonspin conditions and is thus $2g\ \gamma RT_0/(\gamma + 1)$].

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$$V_{a_*}^2 = 2g\ \gamma RT_0/(\gamma + 1) - [(\gamma - 1)/(\gamma + 1)]\omega_0^2 \times (R_0/R_*)^4\, r^2 \quad (3)$$

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As may be seen, the results do not agree. Therefore, it is necessary that the three treatments be examined. First, let us have a brief explanation of this author's treatment of the problem, as follows.

Before going into mathematical analysis of the system, let us look at the physical situation. First, we define a stream tube as in Fig. 1. It may easily be shown that the assumptions of maintenance of a forced vortex and conservation of angular momentum require that the flow contract uniformly through the nozzle to the throat; that is, the ratio of the radial positions of two stream lines is independent of axial position. Thus, it is seen that the subtended cross-sectional area decreases monotonically to the throat and then increases. However, the important point to note is that the flow in the stream tube is not perpendicular to the physical cross-sectional area. Thus, the area seen by the flow (neglecting radial components) is

$$A_{st, perp}\sin\phi = A_{st, perp}\, V_a/(V_a^2 + V_t^2)^{1/2}$$

where $A_{st, perp}$ is defined as the area that a stream tube cuts in a plane perpendicular to the nozzle axis. Therefore, if $V_a/(V_t^2 + V_a^2)^{1/2}$ at some point P upstream of the throat is small enough compared to its value at the physical throat that $A_{st, perp, p}[V_a/(V_a^2 + V_t^2)^{1/2}]_p < A_{st, perp, *}[V_a/(V_a^2 + V_t^2)^{1/2}]_*$, then the resultant velocity is choked upstream of the throat rather than at the throat.

In the mathematical analysis of the flow, the following assumptions were made by this author: 1) Solid body rotation (forced vortex) is maintained as the flow necks down through the nozzle throat. 2) Angular momentum of each particle of fluid is conserved. 3) The gas is ideal with constant specific heat ratio (γ). 4) The flow is frozen (thus there are no heat of reaction effects or changes in molecular weight). 5) Radial velocities are negligibly small. 6) The flow is isentropic along a stream tube.

As mentioned earlier, the first two assumptions constrict the annular stream shells (sets of stream tubes) to taper proportionally to the nozzle hardware; that is,

$$r = r_0(R/R_0) \quad (4)$$

Moreover, the angular velocity (constant with respect to radial position) is given by

$$\omega = \omega_0(R_0/R)^2 \quad (5)$$

Additionally, the area (perpendicular to the motor axis) of a stream tube may be expressed as (see Fig. 1)

$$A_{st, perp} = A_{st, perp, 0}\ (R/R_0)^2 \quad (6)$$

The next step is to write the following equations along a stream tube: a) continuity equation, b) Euler equation, and c) isentropic relationship.

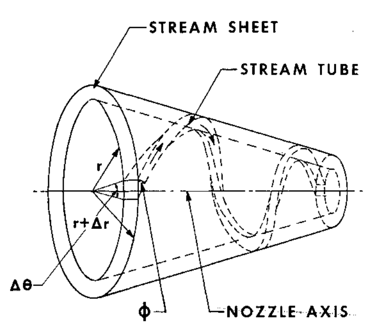

Fig. 1 Stream tube in converging section of nozzle.

Continuity: The stream-tube area perpendicular to the direction of flow may be expressed by

$$A_{st} = A_{st,\ perp}[V_a^2/(V_t^2 + V_a^2)]^{1/2}$$

where $A_{st,\ perp}$ is defined as before. Therefore, the continuity equation may be written two ways (neglecting V_r) as

$$\dot{m}_{st} = \rho A_{st}(V_a^2 + V_t^2)^{1/2} = \rho\, A_{st,\ perp}\, V_a \quad (7)$$

Euler equation along a stream tube: This may be written, neglecting initial velocities in comparison to velocities in the throat region (as done in simplified one-dimensional treatments), and neglecting radial velocity, as

$$\int_{P_0}^{P} \frac{dp}{\rho} + \frac{V_t^2 + V_a^2}{2} = 0 \quad (8)$$

Isentropy:

$$P/\rho^\gamma = \text{const} = K \quad (9)$$

Equations (4, 5, and 7–9) may be combined to yield an expression for $[\dot{m}_{st}/A_{st,\ perp}]$. Since $A_{st,\ perp}$ is minimized at the throat, $[\dot{m}_{st}/A_{st,\ perp}]$ must be maximized there. Application of this criterion (the "choking criterion") by differentiation of the expression for $\dot{m}_{st}\ A_{st,\ perp}$ with respect to pressure and setting the resulting expression equal to zero yields the throat pressure for each streamline,

$$P_* = \left[\left(\frac{2}{\gamma + 1}\right) P_0^{(\gamma-1)/\gamma} - \frac{(\gamma - 1)K^{-1/\gamma}}{(\gamma + 1)\,\gamma}\,\omega_0^2 \left(\frac{R_0}{R^*}\right)^4 r^2\right]^{\gamma/(\gamma-1)} \quad (10)$$

which upon suitable algebraic manipulation yields Eq. (3) for the square of the axial velocity. Still more mathematical exercise shows that the axial Mach number (V_a/a) at the physical throat is 1.0, independent of the radial position while the resultant Mach number $[(V_a^2 + V_t^2)^{1/2}/a]$ passes through 1.0 before the physical throat, everywhere except at the centerline. This results from the fact that, as discussed earlier, flow in a stream tube is not perpendicular to the "hardware" cross-sectional area; and thus the fluid in a given stream tube may see a minimum area perpendicular to the flow direction upstream of the physical throat. The mathematical detail presented here is quite skimpy because of length restrictions, but can be greatly amplified on request.

But now let us examine Eq. (10) a bit more carefully. A small amount of algebraic manipulation or numerical substitution shows that the radial Navier-Stokes equation cannot be satisfied simultaneously in the motor chamber and at the throat. Let us recall that the radial Navier-Stokes equation was not employed in the analysis.

Let us next turn to Manda's analysis.[1] He assumes conservation of angular momentum and proportional tapering of his gas "rings." These assumptions may be shown to be entirely equivalent to the assumptions of conservations of angular momentum and maintenance of an ideal forced vortex pattern. He then uses a form of the equation of conservation of energy that is equivalent to the use of the Euler equation, the isentropy relationship, and the ideal gas law, along each stream tube in combination with the radial Navier-Stokes equation at the throat (using the one-dimensional, nonswirl case to obtain a boundary condition at the centerline since the tangential velocity is zero at the centerline) in order to solve for the axial velocity at the throat, V_{a_*}. In the application of the energy equation, he does not neglect the initial tangential velocity as this author does in application of the Euler equation. This, however, should lead to only a small difference in results and is not important to the discussion. It should be noted that nowhere has the concept of choked flow been brought in and that, in fact, according to the analysis of this author, this criterion has not been satisfied.

Finally, let us look at Bastress' treatment.[2] He has likewise made the assumptions of conservation of angular momentum and maintenance of a "spinning-plug" flow (or, equivalently, uniform contraction of the stream-lines). However, he has additionally assumed the radial pressure gradient to be everywhere zero. Finally, he essentially assumed that the temperature, pressure, and density will be the same at the throat as in the nonspin case when he assumed that the sum of the squares of the axial and tangential velocities in the spin case is equal to the square of the axial velocity in the nonspin case. The final result satisfies neither the radial Navier-Stokes equation nor the "choke criterion."

Conclusions

By now, it should be seen that the solutions have one problem in common; that is, they do not satisfy simultaneously all of the governing equations which must be applied to them. Bastress' system is seriously overspecified; those of Manda and this author are overspecified by only one degree. In the one-dimensional nonswirl case, there are basically five unknowns (T, P, ρ, V_a, $\dot{m}$) and four equations (Euler, continuity, the ideal gas law, and equivalently either the energy equation or the isentropic relationship) plus the choke criterion. When swirling flow is brought in, the additional variables of ω and V_t enter. However, if conservation of angular momentum and uniform tapering of stream sheets (or equivalently, maintenance of a forced vortex pattern) are required, these criteria, along with the radial Navier-Stokes equation, provide three new independent equations. Thus, the problem is overspecified. Therefore, it appears to this author that the assumption of maintenance of a forced vortex flow pattern must be dropped for accurate solution of the problem of flow of a fluid from a chamber in which it is swirling through a converging nozzle.

References

[1] Manda, L., "Spin effects on rocket nozzle performance," J. Spacecraft Rockets **3,** 1695–1696 (1966).

[2] Bastress, E. K., "Interior ballistics of spinning solid-propellant rockets," J. Spacecraft Rockets **2,** 455–457 (1965).

Reply by Author to M. King

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BOTH King's analysis and the author's are attempts to approximate the solution of the inviscid Navier-Stokes equations for axisymmetric subsonic flow with rotation. In both efforts, velocities in the radial direction (V_r) are assumed negligible. Therefore, with axial symmetry ($\partial/\partial\theta = 0$) and $V_r = 0$, the equations of motion in polar coordinates can be reduced to

$$(1/\rho)(\partial P/\partial x) = -V_a(\partial V_a/\partial x) \quad (1)$$

$$(1/\rho)(\partial P/\partial \theta) = 0 = -V_a(\partial/\partial x)(rV_t) \quad (2)$$

$$(1/\rho)(\partial P/\partial r) = (V_t^2/r) \quad (3)$$

which are applicable throughout the flowfield.

In addition, the conservation of mass (in any annular stream-tube) requires that

$$\rho V_a(\partial A/\partial x) + A(\partial/\partial x)(\rho V_a) = 0 \quad (4)$$

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